Free convection in the tilted open thermosyphon

By F. M. LESLIE
Department of Mathematics, University of Manchester

(Received 15 April 1959)

Approximate solutions are found for the fluid flow and heat transfer in a heated cylinder, closed at the bottom and opening at the top into a reservoir of cool fluid, which has been tilted at a small angle to the vertical.

One solution is found for large Rayleigh number when the boundary layer does not fill the tube, and another for small Rayleigh number when the boundary layer fills the tube. In both cases tilting causes a small increase in heat transfer which is proportional to the square of $(l/a)\tan\phi$, where l/a is the length–radius ratio and ϕ the angle of tilt.

1. Introduction

The open thermosyphon consists of a heated vertical cylinder of fluid, closed at the lower end and opening at the top into a reservoir of cool fluid. The fluid is subject to gravity. An annulus of hot fluid at the walls, being less dense, is displaced towards the open end and is replaced by heavier cool fluid moving down the centre of the tube. Thus a circulation is created which transfers heat by convection from the walls of the cylinder to the reservoir.

Holzwarth (1938) suggested that a similar system could be used to cool gas turbine blades. He envisaged a cylindrical cavity in each blade opening into a reservoir of cool fluid in the hub. In this case the external axial acceleration is the centrifugal acceleration, which can be as large as $10^4 g$. Since the system is rotating, there is also a Coriolis acceleration at right angles to the motion of the particles, and hence flow is not quite axially symmetrical.

No attempt had been made to solve this problem theoretically until Lighthill (1953) investigated the action of the thermosyphon in some detail for laminar and turbulent flow. He considered a cylinder whose walls are maintained at a constant temperature, and which had a constant axial acceleration directed towards the lower closed end; and he assumed that the fluid entering the tube along the axis at the open end has the temperature of the reservoir.

In the present paper an attempt is made to extend Lighthill's work by considering the external acceleration to be at a small angle to the axis. Alcock (1951) has pointed out that this problem is similar to that of free convection in rotating turbine blades. The component of the buoyancy force perpendicular to the axis simulates closely the Coriolis force on the moving fluid. The case of Coriolis forces is very difficult to treat, but that of simulated forces can be solved to give a fair estimate of the effect.

Martin (1957) has done experimental work on this problem, using a heated cylinder of fluid closed at the bottom and opening at the top into a reservoir of cool fluid. The external acceleration was gravity, and he investigated the effect of tilting the cylinder from the vertical position. In work so far unpublished, Martin found that, for Rayleigh number $10^{7\cdot25}$, the Nusselt number decreases with increasing angle of inclination to the vertical up to 5°, where it is 90 % of its value for the tube in the vertical position. If the inclination is increased beyond this point, the Nusselt number rises to its initial value by 15° and thereafter to 130 % of this value by 45°. For smaller Rayleigh numbers the initial decrease is smaller, until when the Rayleigh number is $10^{5\cdot25}$ there is no decrease and the Nusselt number increases steadily to 150 % of its initial value at 45°. Martin put forward an explanation for this decrease, which is equivalent to there being an instability at small angles of inclination for larger Rayleigh numbers. He attributed the ensuing rise in heat transfer to an overall decrease in the thickness of the boundary layer which is formed at the walls.

In the present treatment, only laminar flow is considered, and no attempt is made to assess the effects of a breaking down of laminar flow. Attention is given mainly to the flow obtained for large values of Rayleigh number, since a typical value for gas turbine applications is of the order of 10^{10}; however, a solution is also found for small Rayleigh number.

The most important parameter used by Lighthill is $t_1 = (a/l)\,Aa$, where a is the radius of the cylinder and l its length, and Aa is the Rayleigh number based on radius. For large t_1 (> 3400), he found a solution in the form of a rising boundary layer at the walls and a cool descending uniform core in the centre. Because of the complexity of the equations, he used approximate methods of solution based on integrated equations which represent the conservation of mass energy and momentum across a section of the cylinder. For small t_1 (< 311), Lighthill found a similarity solution in which velocity and temperature profiles are similar at different axial positions but vary in magnitude linearly with axial distance.

Solutions of a similar type are found below for the problem of the inclined tube. The boundary-layer thickness is found to decrease, giving an increase in heat transfer, as the inclination of the external acceleration to the axis is increased, which is in agreement with Martin's work. No initial decrease in the Nusselt number is found, however, since the model assumed does not allow for instabilities in the boundary layer in the cylinder due to unstable stratification on the lower side of the tube. This increase in heat transfer varies as the square of $(l/a)\tan\phi$, where ϕ is the angle between the external acceleration and the axis of the cylinder. For large t_1, where the flow is of the boundary-layer type which does not fill the tube, the percentage increase in the Nusselt number is $0\cdot17(l/a)^2\tan^2\phi$; and for small t_1, where the boundary layer fills the tube with similarity, this factor is $25(l/a)^2\tan^2\phi$. It is difficult to compare the results obtained in the present paper with those found by Martin in his experiments on account of the initial decrease in the Nusselt number due to another effect. However, there does appear to be qualitative agreement with his results for smaller Rayleigh numbers where there is no initial decrease. They are also supported to some extent by his results for larger angles of inclination at all Rayleigh numbers.

2. The equations of motion

Cylindrical polar co-ordinates (X, R, θ) are used where the X-axis coincides with the axis of the cylinder, the origin being at the closed end and the positive direction towards the open end. The transverse component of body force is directed towards $\theta = 0$. The equations governing the motion are those of conservation of mass, energy and momentum for steady motion; these are respectively

$$\frac{\partial U}{\partial X}+\frac{1}{R}\frac{\partial}{\partial R}(RV)+\frac{1}{R}\frac{\partial W}{\partial \theta} = 0, \tag{1}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial R}+\frac{W}{R}\frac{\partial T}{\partial \theta} = K\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial X^2}\right], \tag{2}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial R}+\frac{W}{R}\frac{\partial U}{\partial \theta} = -\frac{1}{\rho}\frac{\partial P}{\partial X}-f\cos\phi+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial X^2}\right], \tag{3}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial R}+\frac{W}{R}\frac{\partial V}{\partial \theta}-\frac{W^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R}+f\sin\phi\cos\theta$$
$$+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial X^2}-\frac{V}{R^2}-\frac{2}{R^2}\frac{\partial W}{\partial \theta}\right], \tag{4}$$

$$U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial R}+\frac{W}{R}\frac{\partial W}{\partial \theta}+\frac{VW}{R} = -\frac{1}{\rho}\frac{1}{R}\frac{\partial P}{\partial \theta}-f\sin\phi\sin\theta$$
$$+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{\partial^2 W}{\partial X^2}-\frac{W}{R^2}+\frac{2}{R^2}\frac{\partial V}{\partial \theta}\right], \tag{5}$$

where U, V, W are components of velocity in the directions of X, R, θ; T is temperature, P pressure, ρ density, f the external acceleration making angle ϕ with the negative direction of the X-axis; and K is the thermal diffusivity and γ the kinematic viscosity.

An equation of state is also required; this is taken to be

$$\frac{1}{\rho} = \frac{1}{\rho_0}(1+\alpha(T-T_0)), \tag{6}$$

where α is the coefficient of cubical expansion and the subscript 0 refers to conditions at the wall.

The boundary conditions are

$$\left.\begin{array}{llll} U = V = W = 0 & \text{at} \quad R = a, & \text{or} & X = 0, \\ T = T_0 & \text{at} \quad R = a, & \text{or} & X = 0, \\ T = T_1 & \text{at} \quad R = 0, & & X = l, \end{array}\right\} \tag{7}$$

where T_0 and T_1 are constant temperatures, T_1 being the temperature of the reservoir, and a, l are the radius and length of the cylinder.

Since either the flow is of the boundary-layer type or the ratio of radius to length is small, second derivatives with respect to X are neglected in comparison with those with respect to R. Large Prandtl number is assumed, and as a result the inertia terms in the momentum equations are neglected. This procedure was adopted by Lighthill and led to an error of 5 % when the Prandtl number was five.

Putting $P = P_1 + p$ where p is due to velocity effects, equations (3) to (5) give, when the velocity is zero,

$$\left.\begin{aligned}\frac{\partial P_1}{\partial X}+\rho_0 f\cos\phi &= 0,\\ \frac{\partial P_1}{\partial R}-\rho_0 f\sin\phi\cos\theta &= 0,\\ R\frac{\partial P_1}{\partial\theta}+\rho_0 f\sin\phi\sin\theta &= 0.\end{aligned}\right\}\tag{8}$$

Thus P_1 is the hydrostatic pressure when the temperature is T_0 throughout. Eliminating P_1 from (3) to (5), and using (8) and then (6), we find

$$-\frac{1}{\rho}\frac{\partial p}{\partial X}+\alpha f\cos\phi\,(T-T_0)+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 U}{\partial\theta^2}\right] = 0,\tag{9}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial R}-\alpha f\sin\phi\,(T-T_0)\cos\theta+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 V}{\partial\theta^2}-\frac{V}{R^2}-\frac{2}{R^2}\frac{\partial W}{\partial\theta}\right] = 0,\tag{10}$$

$$-\frac{1}{\rho R}\frac{\partial p}{\partial\theta}+\alpha f\sin\phi\,(T-T_0)\sin\theta+\gamma\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 W}{\partial\theta^2}-\frac{W}{R^2}+\frac{2}{R^2}\frac{\partial V}{\partial\theta}\right] = 0.\tag{11}$$

To reduce equations (1), (2), (9), (10), (11) and boundary equations (7) to non-dimensioned form the following substitutions are made:

$$U = \frac{Kl}{a^2}u^1,\quad V = \frac{K}{a}v^1,\quad W = \frac{K}{a}w^1,$$

$$R = ar,\quad X = lx,$$

$$T-T_0 = -\frac{\gamma Kl}{\alpha f a^4\cos\phi}t^1,\quad p = \frac{\gamma Kl}{a^2}p^1.$$

The equations become

$$\frac{\partial u^1}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv^1)+\frac{1}{r}\frac{\partial w^1}{\partial\theta} = 0,\tag{12}$$

$$u^1\frac{\partial t^1}{\partial x}+v^1\frac{\partial t^1}{\partial r}+\frac{w^1}{r}\frac{\partial t^1}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t^1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 t^1}{\partial\theta^2},\tag{13}$$

$$-\left(\frac{a}{l}\right)^2\frac{\partial p^1}{\partial x}-t^1+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u^1}{\partial\theta^2} = 0,\tag{14}$$

$$-\frac{\partial p^1}{\partial r}+\epsilon t^1\cos\theta+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v^1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v^1}{\partial\theta^2}-\frac{v^1}{r^2}-\frac{2}{r^2}\frac{\partial w^1}{\partial\theta} = 0,\tag{15}$$

$$-\frac{1}{r}\frac{\partial p^1}{\partial\theta}-\epsilon t^1\sin\theta+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w^1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w^1}{\partial\theta^2}-\frac{w^1}{r^2}+\frac{2}{r^2}\frac{\partial v^1}{\partial\theta} = 0,\tag{16}$$

where $\epsilon = (l/a)\tan\phi$. The boundary conditions (7) reduce to

$$\left.\begin{aligned}u^1 = v^1 = w^1 = t^1 = 0 \quad &\text{at}\quad r = 1,\quad\text{or}\quad x = 0,\\ t^1 = t_1 = \alpha f\cos\phi\,(T_0-T_1)\frac{a^4}{\gamma Kl}\quad &\text{at}\quad r = 0,\quad x = 1.\end{aligned}\right\}\tag{17}$$

Since there is symmetry about the plane $\theta = 0$ and the angle ϕ is taken as small, solutions in the form of the leading terms of Fourier series are obtained:

$$u^1 = U+u\cos\theta,\quad v^1 = V+v\cos\theta,\quad w^1 = w\sin\theta,$$

$$p^1 = P+p\cos\theta,\quad t^1 = T+t\cos\theta,$$

where U, V, P, T, and perturbation functions u, v, w, p, t are functions of x, r only. It should be noted that the symbols U, V, P, T, and p have been used earlier with a different meaning.

The boundary conditions (17) become

$$\left.\begin{aligned} U = V = T = 0 \quad \text{at} \quad r = 1, \quad \text{or} \quad x = 0, \\ T = t_1 \quad \text{at} \quad x = 1, \quad r = 0, \end{aligned}\right\} \tag{18}$$

and

$$\left.\begin{aligned} u = v = w = t = 0 \quad \text{at} \quad r = 1, \quad \text{or} \quad x = 0, \\ v+w = 0 \quad \text{at} \quad r = 0, \\ u = p = t = 0 \quad \text{at} \quad r = 0. \end{aligned}\right\} \tag{19}$$

This last condition is necessary because of harmonic variation with θ.

Making these substitutions in (12) to (15) and integrating with respect to θ from 0 to 2π to obtain terms independent of θ, we obtain

$$\left.\begin{aligned} \frac{\partial U}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rV) = 0, \qquad (a) \\ U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial r}+\frac{1}{2}\left(u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial r}-w\frac{t}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \qquad (b) \\ -\left(\frac{a}{l}\right)^2\frac{\partial P}{\partial x}-T+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) = 0, \qquad (c) \\ -\frac{\partial P}{\partial r}+\tfrac{1}{2}\epsilon t+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)-\frac{V}{r^2} = 0. \qquad (d) \end{aligned}\right\} \tag{20}$$

Here $\frac{1}{2}(u\partial/\partial x+v\partial/\partial r-w/r)t$ and $\frac{1}{2}\epsilon t$ are of the second order of smallness since u, v, w, t are perturbations; if these terms are neglected relative to other terms, equations (20) reduce to those used by Lighthill. He used integrated forms of this reduced set of equations (20) to find approximate solutions for U, T subject to boundary conditions (18). Thus, if $\frac{1}{2}(u\partial/\partial x+v\partial/\partial t-w/r)t$ and $\frac{1}{2}\epsilon t$ are neglected, U, T may be regarded as known functions of x, r for all t_1, and then can be found from Lighthill (1953).

Equations for the perturbation functions are found by substituting for u^1, v^1, w^1, p^1, t^1 in equations (12) to (16): equations (12) to (15) are multiplied by $\cos\theta$ and (16) by $\sin\theta$ and then integrated with respect to θ from 0 to 2π to give

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{w}{r} = 0, \tag{21}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}+U\frac{\partial t}{\partial x}+V\frac{\partial t}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)-\frac{t}{r^2}, \tag{22}$$

$$-\left(\frac{a}{l}\right)^2\frac{\partial p}{\partial x}-t+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{u}{r^2} = 0, \tag{23}$$

$$-\frac{\partial p}{\partial r}+\epsilon T+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)-\frac{2}{r^2}(v+w) = 0, \tag{24}$$

$$\frac{p}{r}-\epsilon T+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)-\frac{2}{r^2}(r+w) = 0. \tag{25}$$

Eliminating p between equations (24) and (25),

$$r\epsilon\frac{\partial T}{\partial r}-\frac{\partial^2}{\partial r^2}\left(r\frac{\partial w}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{2}{r}\frac{\partial}{\partial r}(v+w) = 0. \tag{26}$$

No attempt is made to solve these equations as they stand because of their complexity. However, they can be expressed as the conservation of mass, energy and momentum (associated with the secondary flow) across a section bounded by the planes θ and $\theta+d\theta$, x and $x+dx$, and $r = 1$ by multiplying by r and integrating from 0 to 1 with respect to r, thus

$$\frac{d}{dx}\int_0^1 ru\,dr+\int_0^1 w\,dr = 0, \tag{27}$$

$$\frac{d}{dx}\int_0^1 r\,(uT+Ut)\,dr+\int_0^1 wT\,dr = \left(\frac{\partial t}{\partial r}\right)_{r=1}-\int_0^1\frac{t}{r}dr, \tag{28}$$

$$-\int_0^1 rt\,dr+\left(\frac{\partial u}{\partial r}\right)_{r=1}-\int_0^1\frac{u}{r}dr = 0. \tag{29}$$

In (29) the term containing $\partial p/\partial x$ has been neglected, where p is the perturbation pressure; this appears reasonable physically and can be justified mathematically later. The integrated form of equation (26) is

$$2\epsilon\int_0^1 rT\,dr+\left(\frac{\partial^2 w}{\partial r^2}\right)_{r=1}+\left(\frac{\partial v}{\partial r}\right)_{r=1} = 0. \tag{30}$$

The values of equations (21) to (26) at the walls and on the axis are needed to determine the profiles to be used in the similarity solution. These are

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)\right]_{r=1} = 0, \tag{31}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{r=1} = 0, \tag{32}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]+(p)_{r=1} = 0, \tag{33}$$

$$\left.\begin{aligned}\left(U\frac{\partial t}{\partial x}+v\frac{\partial T}{\partial r}\right)_{r=0} &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)-\frac{t}{r^2}\right]_{r=0},\\ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)-\frac{2\,(v+w)}{r^2}\right]_{r=0}+\epsilon(T)_{r=0}-\left(\frac{\partial p}{\partial r}\right)_{r=0} &= 0,\\ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)-\frac{2(v+w)}{r^2}\right]_{r=0}-\epsilon(T)_{r=0}+\left(\frac{p}{r}\right)_{r=0} &= 0.\end{aligned}\right\} \tag{34}$$

Simple radial profiles are assumed for velocity and temperature in finding the solution for the boundary layer which does not fill the tube. No attempt is made to satisfy equations (31) to (34), and as a result higher derivatives such as $(\partial^2 w/\partial r^2)_{r=1}$ in (30) cannot be expected to be accurate. It is possible to eliminate this term using equation (33), giving

$$2\epsilon\int_0^1 rT\,dr-\left(\frac{\partial w}{\partial r}\right)_{r=1}+\left(\frac{\partial v}{\partial r}\right)_{r=1}-(p)_{r=1} = 0. \tag{35}$$

To eliminate $(p)_{r=1}$, we first integrate (24) with respect to r from 0 to 1:

$$(p)_{r=1} = \epsilon \int_0^1 T\,dr + \left[\frac{\partial v}{\partial r}\right]_0^1 - 2\left(\frac{v+w}{r}\right)_{r=0} - \int_0^1 \frac{1}{r}\frac{\partial v}{\partial r}\,dr - \int_0^1 \frac{2}{r}\frac{\partial w}{\partial r}\,dr,$$

and deduce that

$$2\epsilon \int_0^1 rT\,dr - \epsilon \int_0^1 T\,dr + \left(\frac{\partial v}{\partial r}\right)_{r=0} - \left(\frac{\partial w}{\partial r}\right)_{r=1} + 2\left(\frac{v+w}{r}\right)_{r=0} + \int_0^1 \frac{1}{r}\frac{\partial v}{\partial r}\,dr + 2\int_0^1 \frac{1}{r}\frac{\partial w}{\partial r}\,dr = 0. \quad (36)$$

The Nusselt number (based on radius) is given by

$$N_a = \frac{Qa}{k(T_0 - T_1)\,2\pi l a},$$

where Q is the actual rate of heat transfer from the whole wall surface of the tube and k is the thermal conductivity of the fluid.

Thus
$$N_a = \frac{1}{2\pi l(T_0 - T_1)} \int_0^l \int_0^{2\pi} a\left(\frac{\partial T}{\partial R}\right)_{R=a} dX\,d\theta$$
$$= \frac{1}{2\pi t_1} \int_0^1 \int_0^{2\pi} \left(-\frac{\partial t^1}{\partial r}\right)_{r=1} dx\,d\theta.$$
$$= \frac{1}{t_1} \int_0^1 \left(-\frac{\partial T}{\partial r}\right)_{r=1} dx. \quad (37)$$

Also
$$t_1 = \frac{\alpha f \cos\phi\,(T_0 - T_1)\,a^4}{\gamma K l}$$
$$= \left(\frac{a}{l}\right) A_a,$$

where
$$A_a = \frac{\alpha f \cos\phi\,(T_0 - T_1)\,a^3}{\gamma K}$$

is the Rayleigh number based on radius.

When the effect of the perturbations upon the solution obtained by Lighthill is calculated, several of equations (20) are required in modified form retaining the perturbation terms. With the usual boundary layer assumptions (20*d*) reduces to $\partial P/\partial r = 0$, and, at the wall, (20*c*) becomes

$$-\left(\frac{a}{l}\right)^2 \frac{dP}{dx} + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)\right]_{r=1} = 0.$$

This equation is used to eliminate P from (20*c*) which is then integrated over a cross-section; thus

$$-\int_0^1 rT\,dr + \frac{1}{2}\left[\frac{\partial U}{\partial r} - \frac{\partial^2 U}{\partial r^2}\right]_{r=1} = 0. \quad (38)$$

When integrated over a cross-section (20*b*) reduces to

$$\frac{d}{dx}\int_0^1 rUT\,dr + \frac{1}{2}\frac{d}{dx}\int_0^1 rut\,dr = \left(\frac{\partial T}{\partial r}\right)_{r=1}. \quad (39)$$

On the axis (20*b*) takes the form

$$\left[U\frac{\partial T}{\partial x}+\frac{1}{2}\left(v\frac{\partial t}{\partial r}-w\frac{t}{r}\right)\right]_{r=0}=\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]_{r=0}. \tag{40}$$

Equations (38) to (40) with the perturbation terms neglected are in the form used by Lighthill.

3. Boundary-layer solution not filling tube

In this section a solution is found in the form of a rising flow in a boundary layer at the walls and a descending flow in a cool uniform core in the centre.

For equations (20) Lighthill assumed the following solution:

$$\left.\begin{aligned} U &= -\phi && (0<r<\beta),\\ &= -\phi\left[1-\left(\frac{r-\beta}{1-\beta}\right)^2(1+\delta(r-1))\right] && (\beta<r<1),\end{aligned}\right\}$$

$$\left.\begin{aligned} T &= t_1 && (0<r<\beta),\\ &= t_1\left[1-\left(\frac{r-\beta}{1-\beta}\right)^2\right] && (\beta<r<1),\end{aligned}\right\}$$

where ϕ, δ, β are functions of x only, which were determined by substitution into the integrated forms of the equations. The solution obtained by Lighthill can be written in the modified form

$$\delta=\frac{6}{\xi^2},\quad \phi=\frac{t_1\xi^3}{18}$$

and

$$\xi=\left(\frac{240}{t_1}\right)^{\frac{1}{4}}x^{\frac{1}{4}},$$

where $\beta = 1-\xi$ (i.e. ξ is the boundary-layer thickness) and where higher powers of ξ in the expressions for δ and ϕ have been neglected so that ξ may be expressed in terms of x in manageable form. This procedure is accurate to within 2 or 3 % if t_1 is greater than 10^8.

Radial profiles will be chosen for u, v, w, and t as follows:

$$\left.\begin{aligned} u &= 0 && (0<r<\beta),\\ &= \Phi\frac{(1-r)(r-\beta)^2}{(1-\beta)^3} && (\beta<r<1),\end{aligned}\right\}$$

$$\left.\begin{aligned} v &= \gamma && (0<r<\beta),\\ &= \gamma\left[1-\left(\frac{r-\beta}{1-\beta}\right)^2\right] && (\beta<r<1),\end{aligned}\right\}$$

$$\left.\begin{aligned} w &= -\gamma && (0<r<\beta),\\ &= -\gamma\left[1-\left(\frac{r-\beta}{1-\beta}\right)^2(1+\lambda(r-1))\right] && (\beta<r<1),\end{aligned}\right\}$$

$$\left.\begin{aligned} t &= 0 && (0<r<\beta),\\ &= \Theta\frac{(1-r)(r-\beta)^2}{(1-\beta)^3} && (\beta<r<1),\end{aligned}\right\}$$

where Φ, γ, λ and Θ are functions of x only. It is assumed that the boundary-layer thickness remains independent of θ despite the tilting. Martin (in a paper to be published) found very little variation of boundary-layer thickness with θ in his experiments.

If equations (27), (29), (28), (36) are written in the form of power series in ξ and only the lowest power of ξ is retained in each term, then we have

$$\frac{d}{dx}(\Phi\xi) - \gamma(12 + \lambda\xi^2) = 0.$$

$$12\Phi + \xi^2\Theta = 0,$$

$$\frac{d}{dx}\left[\frac{t_1\Phi\xi}{20} + \frac{t_1\Theta\xi^3}{315}\right] - \gamma t_1\left(1 + \frac{\lambda\xi^2}{20}\right) = -\frac{\Theta}{\xi},$$

$$\gamma\lambda + \frac{\epsilon t_1\xi}{3} = 0.$$

Solving for Φ, Θ, λ and γ to the lowest order in ξ, we find

$$\Phi = -\frac{\epsilon t_1^2\xi^6}{84 \times 45}; \quad \Theta = \frac{\epsilon t_1^2\xi^4}{7 \times 45};$$

$$\gamma = \frac{\epsilon t_1\xi^3}{54}; \quad \lambda = -\frac{18}{\xi^2}.$$

Figure 1*a* shows the streamlines in the plane of symmetry ($\theta = 0$) for $t_1 = 10^8$ and $\epsilon = 1$. It can be seen that fluid in the cool uniform core is drawn to one side of the cylinder by the component of external acceleration in that direction. In the boundary layer the fluid tends to flow around the circumference of the cylinder due to this component of the buoyancy force. Figure 1*b*, which shows the projections of the streamlines on to a cross-section of the cylinder, also illustrates the cool fluid moving towards one side of the cylinder.

With these results equation (39) may now be written

$$\frac{d}{dx}\left[\frac{t_1\xi^3}{180} + \frac{\epsilon^2 t_1^{\frac{1}{4}}}{(240)^{\frac{1}{4}}}\frac{256}{27 \times 7^3}x^{\frac{11}{4}}\right] = \frac{1}{\xi}, \tag{41}$$

where again only the leading power in ξ is retained in the first term in the bracket. The value of ξ as a function of x quoted above is calculated from this equation where the second term in the bracket (due to perturbations) has been neglected, and hence is accurate only if $\epsilon^2x^2 \ll 13$. To find the effect of the perturbations upon the solution obtained by Lighthill, equation (41) with the perturbation term retained is solved for ξ as a function of x by writing $\xi = (240/t_1)^{\frac{1}{4}}x^{\frac{1}{4}}(1+\eta)$ and neglecting squares and higher powers of η. It is found that $\eta = -0{\cdot}0063\epsilon^2x^2$. If it is noted that, from (37),

$$N_a = 2\int_0^1 \frac{1}{\xi}\,dx,$$

it follows that the change in the Nusselt number is given by

$$N_a = \frac{8}{3}\left(\frac{t_1}{240}\right)^{\frac{1}{4}}(1 + 0{\cdot}0017\epsilon^2).$$

Thus the secondary flow leads to a percentage increase of $0{\cdot}17\epsilon^2$ in the Nusselt number. It can be seen that the increase in heat transfer is of the second order of small quantities taking the perturbations to be of first order of small quantities. The accuracy claimed for this perturbation solution is in the region of 25 % provided that t_1 is greater than 10^8, ϵ is less than unity, and the Prandtl number is of order 10 or greater.

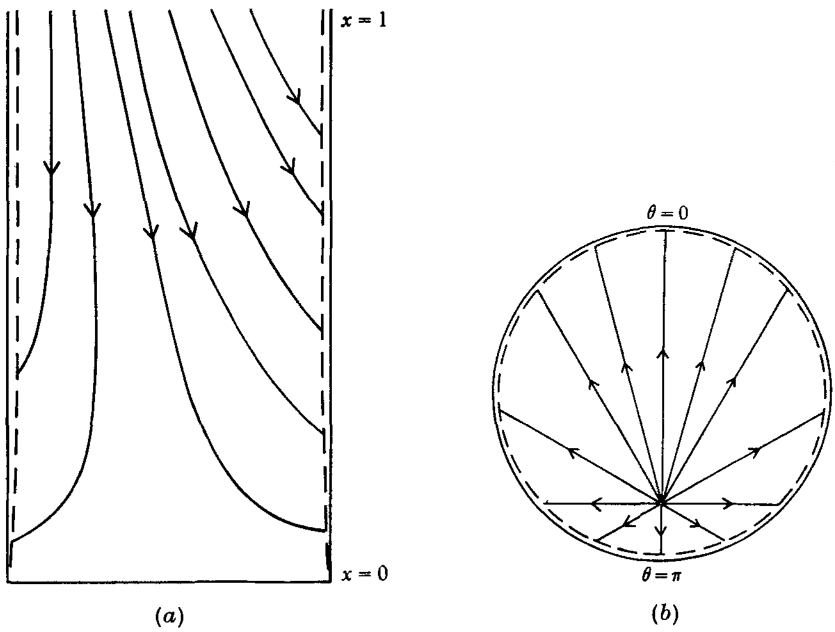

FIGURE 1. (*a*) Sketch of streamlines in plane of symmetry ($\theta = 0$) for $t_1 = 10^8$ and $\epsilon = 1$, i.e. for boundary-layer flow. (*b*) Sketch of projections of streamlines on to a cross-section of the cylinder ($x = 0{\cdot}75$) for $t_1 = 10^8$ and $\epsilon = 1$. --- Denotes edge of boundary layer.

This calculation was also carried out for the profiles as before except that

$$u = \Phi\,\frac{(1-r)\,(r-\beta)}{(1-\beta)^2} \quad (\beta < r < 1),$$

and

$$t = \Theta\,\frac{(1-r)\,(r-\beta)}{(1-\beta)^2} \quad (\beta < r < 1).$$

This had the effect of altering Θ, λ, and γ by a factor of 2 approximately. The corresponding value of η was $-0{\cdot}012\epsilon^2x^2$ and the percentage increase in N_a was $0{\cdot}32\epsilon^2$. This may appear to be rather a large change for the alteration of the profiles but the approximate nature of the solutions must be borne in mind.

Under the conditions of operation of a gas turbine ϵ is of order unity. Hence this calculation suggests that the Coriolis force gives rise to an increase in heat transfer of order 1 % when the flow is laminar.

4. The similarity solution

Lighthill found approximate solutions to equations (20) such that the dependence of U, T on r remains the same although an amplitude factor varies with x, the distance from the bottom of the tube. These solutions are

$$T = t_1 x\left(1-\beta r^2+\frac{(8\beta-9)}{5}r^4+\frac{(4-3\beta)}{5}r^6\right),$$

$$U = -4\beta x(1-6r^2+9r^4-4r^6)-\frac{t_1 x}{24}(r^2-3r^4+2r^6),$$

where the Prandtl number has been assumed infinite. Lighthill found

$$t_1 = 311 \quad \text{and} \quad \beta = 2{\cdot}091.$$

If solutions of a similar type exist for the perturbations then it can be seen from equation (30) that w and v must vary with x and so from (27) that u varies with x^2 and from (29) that t must also vary with x^2. Equations (31), (32), (34) and (30) with boundary conditions (19) are satisfied by

$$u = \lambda x^2 r^4(7-12r^2+5r^4),$$

$$t = \xi x^2 r^4(7-12r^2+5r^4),$$

$$v = x\left[\phi+\left(\frac{2\phi}{3}-\frac{\epsilon t_1}{720}(7\beta+264)\right)r^2+\left(-\frac{5\phi}{3}+\frac{\epsilon t_1}{720}(7\beta+264)\right)r^4\right],$$

$$w = x\left[-\phi+\left(\frac{2\phi}{3}+\frac{\epsilon t_1}{720}(96-7\beta)\right)r^2+\left(\frac{\phi}{3}-\frac{\epsilon t_1}{720}(96-7\beta)\right)r^4\right],$$

where λ, ξ and ϕ are constants to be determined.

From equation (27), we get

$$15\lambda-32\phi+6b = 0,$$

where $$b = \frac{et_1}{720}(96-7\beta).$$

Also from (29), $$4\xi+105\lambda = 0;$$

and from (28) $$\lambda t_1\frac{(796-197\beta)}{2800}+\xi\left(\frac{2\beta}{7}+\frac{35}{8}+\frac{t_1}{3360}\right)$$

$$-\phi t_1\frac{(44259-559\beta)}{7\times 11\times 25\times 27}+t_1\frac{26(388-96\beta)}{3\times 5\times 7\times 9\times 11} = 0.$$

These equations give

$$\phi = 3{\cdot}7\epsilon, \quad \lambda = -6{\cdot}2\epsilon, \quad \xi = 160\epsilon.$$

Figure 2*a* shows a sketch of the streamlines in the plane of symmetry. It can be seen that there is a tendency for the cool fluid on the axis to be drawn to one side of the cylinder by the component of external acceleration in that direction. Also the hot fluid near the walls is influenced by this component of buoyancy force and at one side moves nearer the wall and at the other moves away from the wall. Figure 2*b*, which shows the projections of the streamlines on to a cross-section of the cylinder, indicates clearly the fluid moving away from the wall of the cylinder at one side. It also shows the hot fluid moving around the circumference of the cylinder under the influence of the non-axial component of buoyancy force.

After he obtained the profiles stated above for U, T, Lighthill found t_1 and β from (38) and (39) with the perturbation term in (39) omitted. Thus for the values of U, T found by Lighthill to be valid the second term on the left-hand side of (39) must be negligible. However, if this term is retained to find the change in β and

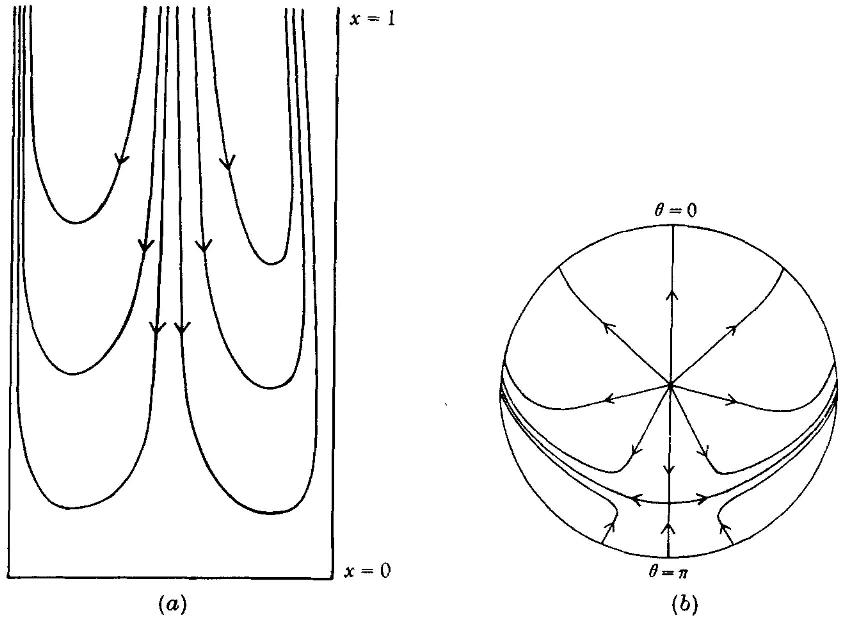

FIGURE 2. (a) Sketch of streamlines in plane of symmetry ($\theta = 0$) for $t_1 = 311$ and $\epsilon = 0{\cdot}03$, i.e. for similarity flow. (b) Sketch of projections of streamlines on to a cross-section of the cylinder ($x = 1{\cdot}0$) for $t_1 = 311$ and $\epsilon = 0{\cdot}03$.

t_1 due to the presence of those perturbations, it can be seen that β and t_1 now depend upon x. Using (40), equations (39) and (38) become

$$\frac{d}{dx}\left[\frac{\beta t_1 x^2(2\beta+9)}{\left(1+\dfrac{x}{t_1}\dfrac{dt_1}{dx}\right)105}+\frac{t_1^2x^2}{4200}\frac{13-\beta}{8}\right]$$
$$+\frac{d}{dx}\left[\tfrac{47}{1260}\times 6{\cdot}2\times 160\epsilon^2x^4\right]=\frac{t_1x}{5}(12-4\beta),$$

and
$$t_1\frac{(24+7\beta)}{120}=\frac{48\beta}{\left(1+\dfrac{x}{t_1}\dfrac{dt_1}{dx}\right)},$$

where t_1 and β vary with x. The solutions obtained by Lighthill are permissible for U, T if $\epsilon^2x^2 \ll 3$, since then the second term on the left-hand side of (39) is negligible. The two equations are solved to give

$$t_1 = 311(1-0{\cdot}04\epsilon^2x^2), \qquad \beta = 2{\cdot}091-0{\cdot}38\epsilon^2x^2,$$

and thus
$$Na = 0{\cdot}364+0{\cdot}09\epsilon^2.$$

In this case the percentage increase in the Nusselt number is $25\epsilon^2$. The accuracy of the perturbation solution is difficult to assess but cannot be expected to be more than 20 % on account of the approximate methods used.

However, for both ranges of t_1 investigated, it can be seen that there is an increase in heat transfer when the external acceleration makes a small angle with the axis of the cylinder and it appears reasonable to assume that this is so for the complete range of t_1.

My thanks are due to Prof. Lighthill and Mr B. R. Morton for their many helpful suggestions and to Dr B. W. Martin for allowing me to quote results which have not yet been published. I am also indebted to the Sir James Caird's Travelling Scholarship Trust whose grant has made this work possible.

REFERENCES

ALCOCK, J. F. 1951 *General Discussion on Heat Transfer, Instn Mech. Engrs, Lond.*, section IV, p. 378.

HOLZWARTH, H. 1938 Die Entwicklung der Holzwarth-Gas Turbine, Holzwarth-Gasturbinen G. m. b. H., Muehlheim-Ruhn.

LIGHTHILL, M. J. 1953 *Quart. J. Mech. Appl. Math.* **6**, 398.

MARTIN, B. W. & CRESSWELL, D. J. 1957 *Engineer*, **204**, 926.